数理解析研究所講究録
714巻 1990年 8-10

# A SUFFICIENT CONDITION FOR UNIVALENCY

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1. Introduction. Sakaguchi [1] proved the following theorem:

THEOREM A. If $f(z)=a_1z+\cdots$, $a_1\neq 0$, is analytic in the open unit disk and satisfies

$$(1.1)\qquad \operatorname{Re}\frac{zf'(z)}{f(z)-f(-z)}>0,\quad |z|<1,$$

then $f(z)$ is univalent and close-to-convex there.

The condition (1.1) means that for every $r$ in $0<r<1$ the point $f(-z)$, $z=re^{i\theta}$, is in the left-hand side of the directional tangent at the point $f(z)$ of the image curve of the circle $\{z: |z|=r\}$ under the function $f(z)$. This fact follows from Lemma of Section 2. The function which satisfies the hypotheses of Theorem A is said to be starlike with respect to the images of symmetrical points (or with respect to symmetrical points).

In this note we shall extend Theorem A.

2. Lemma and definition.

We denote by $U$ the open unit disk $\{z: |z|<1\}$ and by $C_r$ the circle $\{z: |z|=r\}$. Further we denote by $C_\phi(r)$ the image curve of $C_r$ under a function $\phi(z)$.

LEMMA. Let $\phi(z)$ be analytic in $U$, and let $z$ move on $C_r$ in the positive direction. Then a necessary and sufficient condition for a point $w_0$ to lie in the left-hand side of the directional tangent of $C_\phi(r)$ at the point $\phi(z_1)$, $|z_1|=r$, is that

$$(2.1)\qquad \operatorname{Re}\frac{z_1\phi'(z_1)}{\phi(z_1)-w_0}>0$$

holds.

PROOF. The following inequality (2.2) is evidently such a necessary and sufficient condition.

$$(2.2)\qquad 0<\arg\frac{[d\phi(z)]_{z=z_1}}{\phi(z_1)-w_0}<\pi,\quad |z|=r.$$

Rewriting, (2.2) becomes

$$(2.3)\qquad 0<\arg\frac{iz_1\,\phi'(z_1)}{\phi(z_1)-w_0}<\pi,$$

which is equivalent to (2.1).

DEFINITION. Let both $f(z)=a_1z+\cdots$, $a_1\neq 0$, and $g(z)=b_1z+\cdots$, $b_1\neq 0$, be analytic in U. Let z move on $C_r$ in the positive direction. Then it follows from Lemma that a necessary and sufficient condition for the point g(z) to lie in the left-hand side of the directional tangent of $C_f(r)$ at f(z) for every r in $0<r<1$ and for every z on $C_r$ is that

$$(2.4)\qquad \mathrm{Re}\,\frac{zf'(z)}{f(z)-g(z)}>0,\quad z\in U,$$

holds. In this case let us say f(z) to be starlike with respect to the function g(z). Therefore the function f(z) in Theorem A may be called starlike with respect to the function f(-z).

3. Main result.

As stated in Theorem A, a function f(z) which is starlike with respect to f(-z) is univalent. However in general, functions f(z) are not necessarily univalent even if they are starlike with respect to some functions. Concerning this kind of problem we have the following theorem.

THEOREM. Let both $f(z)=a_1z+\cdots$, $a_1\neq 0$, and $g(z)=b_1z+\cdots$, $b_1\neq 0$, be analytic in U. If f(z) and g(z) are starlike each other with respect to g(z) and f(z) respectively, then both f(z) and g(z) are univalent and close-to-convex in U.

PROOF. From the hypotheses we have

$$(3.1)\qquad \mathrm{Re}\,\frac{zf'(z)}{f(z)-g(z)}>0,\quad z\in U,$$

$$(3.2)\qquad \mathrm{Re}\,\frac{zg'(z)}{g(z)-f(z)}>0,\quad z\in U.$$

Hence we have

$$\text{(3.3)} \qquad \mathrm{Re}\, \frac{z(f'(z)-g'(z))}{f(z)-g(z)} > 0, \quad z \in U.$$

On the other hand we see from (3.1) that $z/(f(z)-g(z))$ has no pole in U, so that $a_1 - b_1 \neq 0$. Therefore the function $f(z)-g(z)=(a_1-b_1)z+\cdots$ is univalently starlike with respect to the origin, and so is $g(z)-f(z)$ also. Hence (3.1) and (3.2) show respectively that $f(z)$ and $g(z)$ are univalent and close-to-convex in U.

REMARK 1. Let $g(z)=f(-z)$ in this Theorem. Then the conditions (3.1) and (3.2) become equivalent each other, and so in this case our present Theorem reduces to Theorem A.

REMARK 2. We can consider various univalent functions $f(z)$ by adopting various functions as $g(z)$ in this Theorem. For instance the author is interested in the case that $g(z)=f(cz)$ or $g(z)=cz$.

## REFERENCES

[1] K. Sakaguchi, On certain univalent mapping, J. Math. Soc. Japan, 11 (1959), 72-75.

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